KIMS-2003-07-07

Does Church-Kleene ordinal ω_1^{CK} exist?

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July 7, 2003

Abstract: A question is proposed if a nonrecursive ordinal, the so-called Church-Kleene ordinal ω_1^{CK} really exists.

We consider the systems $S^{(\alpha)}$ defined in [2].

Let $\tilde{q}(\alpha)$ denote the Gödel number of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$, if the Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined.

By "recursive ordinals" we mean those defined by Rogers [4]. Then that α is a recursive ordinal means that $\alpha < \omega_1^{CK}$, where ω_1^{CK} is the Church-Kleene ordinal.

Lemma. The number $\tilde{q}(\alpha)$ is recursively defined for countable recursive ordinals $\alpha < \omega_1^{CK}$. Here 'recursively defined' means that $\tilde{q}(\alpha)$ is defined inductively starting from 0.

Remark. The original meaning of 'recursive' is 'inductive.' The meaning of the word 'recursive' in the following is the one that matches the spirit of Kleene [3] (especially, the spirit of the inductive construction of metamathematical predicates described in section 51 of [3]).

Proof. The well-definedness of $\tilde{q}(0)$ is assured by Rosser-Gödel theorem as explained in [2].

We make an induction hypothesis that for each $\delta < \alpha$, the Gödel number $\tilde{q}(\gamma)$ of the formula $A_{(\gamma)}$ $(= A_{q^{(\gamma)}}(\mathbf{q}^{(\gamma)})$ or $\neg A_{q^{(\gamma)}}(\mathbf{q}^{(\gamma)}))$ with $\gamma \leq \delta$ is recursively defined for $\gamma \leq \delta$.

We want to prove that the Gödel number $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \leq \alpha$.

i) When $\alpha = \delta + 1$, by induction hypothesis we can determine recursively whether or not a given formula A_r with Gödel number r is equal to one of the axiom formulas $A_{(\gamma)}$ $(\gamma \leq \delta)$ of $S^{(\alpha)}$. In fact, we have only to see, for a finite number of γ's with $\tilde{q}(\gamma) \leq r$ and $\gamma \leq \delta$, if we have $A_{(\gamma)} = A_r$ or not. By induction hypothesis that $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \leq \delta$, this is then decided recursively.

Thus Gödel predicate $\mathbf{A}^{(\alpha)}(a, b)$ and Rosser predicate $\mathbf{B}^{(\alpha)}(a, c)$ with superscript α are recursively defined, and hence are numeralwise expressible in $S^{(\alpha)}$. Then the Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined, and the Gödel number $\tilde{q}(\alpha)$ of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$ is defined recursively. Thus $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \leq \alpha$.

ii) If α is a countable *recursive* limit ordinal, then there is an increasing sequence of recursive ordinals $\alpha_n < \alpha$ such that

$$\alpha = \bigcup_{n=0}^{\infty} \alpha_n. \tag{1}$$

In the system $S^{(\alpha)}$, the totality of the added axioms $A_{(\gamma)}$ $(\gamma < \alpha)$ is the sum of the added axioms $A_{(\gamma)}$ $(\gamma < \alpha_n)$ of $S^{(\alpha_n)}$. By induction hypothesis, $\tilde{q}(\gamma)$ is recursively defined for $\gamma < \alpha_n$. Thus in each $S^{(\alpha_n)}$ we can determine recursively whether or not a given formula A_r is an axiom of $S^{(\alpha_n)}$ by seeing, for a finite number of γ's with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha_n$, if $A_{(\gamma)} = A_r$ or not.

This is extended to $S^{(\alpha)}$. To see this, we have only to see the γ's with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$, and determine for those finite number of γ's if $A_{(\gamma)} = A_r$ or not. By (1),

$$\tilde{q}(\gamma) \leq r \ \text{ and } \ \gamma < \alpha \Leftrightarrow \exists n \ \text{ such that } \tilde{q}(\gamma) \leq r \ \text{ and } \ \gamma < \alpha_n.$$

Then by induction on n with using the result in the above paragraph for $S^{(\alpha_n)}$ and noting that the bound r on $\tilde{q}(\gamma)$ is uniform in n, we can show that the condition whether or not $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$ is recursively determined. Whence the question whether or not a given formula A_r is one of the axioms $A_{(\gamma)}$ of $S^{(\alpha)}$ with $\tilde{q}(\gamma) \leq r$ and $\gamma < \alpha$ is determined recursively. Thus Gödel predicate $\mathbf{A}^{(\alpha)}(a, b)$ and Rosser predicate $\mathbf{B}^{(\alpha)}(a, c)$ with superscript α are recursively defined, and hence are numeralwise expressible in $S^{(\alpha)}$. Therefore the Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined, and the Gödel number $\tilde{q}(\alpha)$ of Rosser formula or its negation $A_{(\alpha)}$ ($= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$) is defined recursively. Thus $\tilde{q}(\gamma)$ is recursively well-defined for $\gamma \le \alpha$. This completes the proof of the lemma.

Assume now that α is a countable limit ordinal such that there is an increasing sequence of recursive ordinals $\alpha_n < \alpha$ with

$$\alpha = \bigcup_{n=0}^{\infty} \alpha_n. \tag{2}$$

An actual example of such an α is the Church-Kleene ordinal ω_1^{CK}.

In the system $S^{(\alpha)}$, the totality of the added axioms $A_{(\gamma)}$ ($\gamma < \alpha$) is the sum of the added axioms $A_{(\gamma)}$ ($\gamma < \alpha_n$) of $S^{(\alpha_n)}$. By the lemma, $\tilde{q}(\gamma)$ is recursively defined for $\gamma < \alpha_n$. Thus in each $S^{(\alpha_n)}$ we can determine recursively whether or not a given formula A_r is an axiom of $S^{(\alpha_n)}$ by seeing, for a finite number of γ's with $\tilde{q}(\gamma) \le r$ and $\gamma < \alpha_n$, if $A_{(\gamma)} = A_r$ or not.

This is extended to $S^{(\alpha)}$. To see this, we have only to see the γ's with $\tilde{q}(\gamma) \le r$ and $\gamma < \alpha$, and determine for those finite number of γ's if $A_{(\gamma)} = A_r$ or not. By (2),

$$\tilde{q}(\gamma) \le r \ \text{ and } \ \gamma < \alpha \Leftrightarrow \exists n \ \text{ such that } \tilde{q}(\gamma) \le r \ \text{ and } \ \gamma < \alpha_n.$$

Then by induction on n with using the above result for $S^{(\alpha_n)}$ in the preceding paragraph and noting that the bound r on $\tilde{q}(\gamma)$ is uniform in n, we can show that the condition whether or not $\tilde{q}(\gamma) \le r$ and $\gamma < \alpha$ is recursively determined. Then within those finite number of γ's with $\tilde{q}(\gamma) \le r$ and $\gamma < \alpha$, we can decide recursively if for some $\gamma < \alpha$ with $\tilde{q}(\gamma) \le r$, we have $A_r = A_{(\gamma)}$ or not. Therefore we can determine recursively whether or not a given formula A_r is an axiom of $S^{(\alpha)}$.

Therefore Gödel predicate $\mathbf{A}^{(\alpha)}(a,b)$ and Rosser predicate $\mathbf{B}^{(\alpha)}(a,c)$ are recursively defined, and hence are numeralwise expressible in $S^{(\alpha)}$. Then the Gödel number $q^{(\alpha)}$ of the formula

$$\forall b[\neg A^{(\alpha)}(a,b) \vee \exists c(c \le b \,\&\, B^{(\alpha)}(a,c))]$$

is well-defined, and hence Rosser formula $A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ is well-defined and Rosser-Gödel theorem applies to the system $S^{(\alpha)}$. Therefore we can extend

$S^{(\alpha)}$ consistently by adding one of Rosser formula or its negation $A_{(\alpha)}$ $(= A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)})$ or $\neg A_{q^{(\alpha)}}(\mathbf{q}^{(\alpha)}))$ to the axioms of $S^{(\alpha)}$ and get a consistent system $S^{(\alpha+1)}$.

In particular if we assume a least nonrecursive ordinal ω_1^{CK} exists and take $\alpha = \omega_1^{CK}$, we get a consistent system $S^{(\omega_1^{CK}+1)}$. This contradicts the case ii) of the theorem in [2]. We now arrive at

Question. The least nonrecursive ordinal, the so-called Church-Kleene ordinal ω_1^{CK} has been assumed to give a bound on recursive construction of formal systems (see [1], [5], [6]). However the above argument seems to question if ω_1^{CK} really exists in usual set theoretic sense. How should we think?

References

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